

A NOTE ON THE COEFFICIENTS OF RAWNSLEY'S EPSILON FUNCTION OF CARTAN-HARTOGS DOMAINS

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ABSTRACT. We extend the result of Z. Feng and Z. Tu in [5] by showing that if one of the coefficients a_j , $2 \leq j \leq n$, of Rawnsley's epsilon function associated to a n -dimensional Cartan-Hartogs domain is constant, then the domain is biholomorphically equivalent to the complex hyperbolic space.

1. INTRODUCTION AND STATEMENT OF MAIN RESULT

Consider an n -dimensional complex manifold (M, g) endowed with a Kähler metric g and assume that there exists a globally defined Kähler potential $\varphi : M \rightarrow \mathbb{R}$ for g , i.e. if ω is the Kähler form associated to g , we have $\omega = \frac{i}{2} \partial \bar{\partial} \varphi$. Let \mathcal{H}_α be the weighted Bergman space of square integrable holomorphic functions on M with respect to the measure $e^{-\alpha \varphi} \frac{\omega^n}{n!}$, i.e.:

$$\mathcal{H}_\alpha = \left\{ f \in \text{Hol}(M) \mid \int_M e^{-\alpha \varphi} |f|^2 \frac{\omega^n}{n!} < \infty \right\}.$$

If $\mathcal{H}_\alpha \neq \{0\}$, choose an orthonormal basis $\{f_j\}$ with respect to the product:

$$(f, h)_\alpha = \int_M e^{-\alpha \varphi} f \bar{h} \frac{\omega^n}{n!},$$

and denote by $K_\alpha(x, y)$ the reproducing kernel of \mathcal{H}_α , namely:

$$K_\alpha(x, y) = \sum_j f_j(x) \bar{f}_j(y), \quad x, y \in M.$$

Define the ϵ -function associated to g to be the function:

$$\epsilon_{\alpha g}(x) = e^{-\alpha \varphi(x)} K_\alpha(x, x), \quad x \in M.$$

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In the literature the function $\epsilon_{\alpha g}$ was first introduced by J. Rawnsley under the name of η -function in [7] and later as θ -function in [2].

We say that $\epsilon_{\alpha g}$ admits the *Engliš expansion*:

$$\epsilon_{\alpha g}(x) \sim \sum_{j=0}^{\infty} a_j(x) \alpha^{n-j}, \quad x \in M, \quad (1)$$

for $\alpha \rightarrow +\infty$, if for every integers l, r and every compact $H \subseteq M$,

$$\|\epsilon_{\alpha}(x) - \sum_{j=0}^l a_j(x) \alpha^{n-j}\|_{C^r} \leq \frac{C(l, r, H)}{\alpha^{l+1}}, \quad (2)$$

for some constant $C(l, r, H) > 0$. Such expansion is the counterpart for non-compact manifolds of the celebrated TYZ (Tian-Yau-Zelditch) expansion of Kempf's distortion function for polarized compact Kähler manifolds (see [9] and also [1]). In [4] M. Engliš proved that each of the coefficients $a_j(x)$ in (1) is a polynomial of the curvature of the metric g and its covariant derivatives at x , which can be found by finitely many steps of algebraic operations, and gives an explicit expression of the coefficients a_j for $j \leq 3$.

In this paper we consider the case of Cartan-Hartogs domains, which are defined as follows. Consider a Cartan domain $\Omega \subset \mathbb{C}^d$, i.e. an irreducible bounded symmetric domain, of rank r and numerical invariants a, b . Recall that the triple $\{r, a, b\}$ uniquely determines Ω and in particular it defines the dimension $d = \frac{r(r-1)}{2}a + rb + r$ and the genus $\gamma = (r-1)a + b + 2$ of Ω . Let $K(z, z)$ be the Bergman kernel of Ω and $N_{\Omega}(z, z)$ its *generic norm*, i.e.

$$N_{\Omega}(z, z) = (V(\Omega)K(z, z))^{-\frac{1}{\gamma}}, \quad (3)$$

where $V(\Omega)$ is the total volume of Ω with respect to the Euclidean measure of the ambient complex Euclidean space.

For all positive real numbers μ , a Cartan-Hartogs domains is given by $(M_{\Omega}^{d_0}(\mu), g(\mu))$ where:

$$M_{\Omega}^{d_0}(\mu) = \left\{ (z, w) \in \Omega \times \mathbb{C}^{d_0}, \quad \|w\|^2 < N_{\Omega}(z, z)^{\mu} \right\}, \quad (4)$$

and $g(\mu)$ is the Kähler metric whose associated Kähler form $\omega(\mu)$ can be described by the (globally defined) Kähler potential centered at the origin:

$$\Phi(z, w) = -\log(N_{\Omega}(z, z)^{\mu} - \|w\|^2). \quad (5)$$

The domain Ω is called the *base* of the Cartan-Hartogs domain $M_{\Omega}^{d_0}(\mu)$ (one also says that $M_{\Omega}^{d_0}(\mu)$ is based on Ω). These domains have been considered

by several authors (see e.g. [8] and references therein). In [8] it is shown that for $\mu_0 = \gamma/(d+1)$, $(M_\Omega^1(\mu_0), g(\mu_0))$ is a complete Kähler-Einstein manifold which is homogeneous if and only if Ω is the complex hyperbolic space. In [6] the authors of the present paper proved that for $\Omega \neq \mathbb{CH}^d$, the metric $\alpha g(\mu)$ on $M_\Omega^1(\mu)$ is projectively induced for all positive real number $\alpha \geq \frac{(r-1)a}{2\mu}$, exhibiting the first example of complete, noncompact, nonhomogeneous and projectively induced Kähler-Einstein metric. In [10] the author of the present paper proved that for $d_0 = 1$, $g(\mu)$ is extremal (in the sense of Calabi [3]) if and only if it is Kähler-Einstein and that the coefficient a_2 of Engliš expansion of the ε -function associated to a Cartan-Hartogs domain $(M_\Omega^1(\mu), g(\mu))$ is constant, then $(M_\Omega^1(\mu), g(\mu))$ is Kähler-Einstein, conjecturing also that the converse was true. In [5], Z. Feng and Z. Tu generalize that theorem to generic d_0 and proved that conjecture. More precisely, they prove the following:

Theorem 1 (Z. Feng, Z. Tu [5, Th. 1.3]). *The coefficient a_2 of the Rawnsley's ε -function expansion is a constant on $M_\Omega^{d_0}(\mu)$ if and only if $(M_\Omega^{d_0}(\mu), g(\mu))$ is biholomorphically isometric to the complex hyperbolic space $(\mathbb{CH}^{d+d_0}, g_{\text{hyp}})$.*

Notice that g_{hyp} denotes the hyperbolic metric on \mathbb{CH}^{d+d_0} and

$$(\mathbb{CH}^{d+d_0}, g_{\text{hyp}}) = (M_{\mathbb{CH}^d}^{d_0}(1), g(1)).$$

The prove of the previous theorem is based on the explicit formula for the ε -function $\varepsilon_{\alpha g(\mu)}$ of Cartan-Hartogs domains:

$$\varepsilon_{\alpha g(\mu)}(z, w) = \frac{1}{\mu^d} \sum_{k=0}^d \frac{D^k \tilde{\chi}(d)}{k!} \left(1 - \frac{\|w\|^2}{N_\Omega(z, z)^\mu}\right)^{d-k} \frac{\Gamma(\alpha - d + k)}{\Gamma(\alpha - d - d_0)}, \quad (6)$$

for

$$D^k \tilde{\chi}(d) = \sum_{j=0}^k \binom{k}{j} (-1)^j \tilde{\chi}(d-j)$$

and

$$\tilde{\chi}(d-j) = \prod_{j=1}^r \frac{\Gamma(\mu(d-j) - \gamma - (j+1)\frac{a}{2} + 2 + b + ra)}{\Gamma(\mu(d-j) - \gamma + 1 + (j-1)\frac{a}{2})},$$

where Γ is the usual Γ -function. Observe that formula (6) shows that Engliš expansion of the ε -function of Cartan-Hartogs domains is finite. In [11] the author of this paper uses this formula to prove the existence of a Berezin-Engliš quantization for Cartan-Hartogs domains.

The aim of this paper is to generalize Theorem 1 above (see next section) by proving the following:

Theorem 2. *For all $j = 2, \dots, d + d_0$, any coefficient a_j of the Rawnsley's ϵ -function expansion is a constant on $M_\Omega^{d_0}(\mu)$ if and only if $(M_\Omega^{d_0}(\mu), g(\mu))$ is biholomorphically isometric to the complex hyperbolic space $(\mathbb{C}H^{d+d_0}, g_{\text{hyp}})$.*

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2. PROOF OF THEOREM 2

Due to Theorem 1 we need only to prove that if a_j is constant for some $j = 3, 4, \dots, d + d_0$, then a_2 is.

Consider first the polynomial $P(\alpha)$ in the variable α :

$$P(\alpha) = \frac{\Gamma(\alpha - d + k)}{\Gamma(\alpha - d - d_0)},$$

and observe that for $k = d$, ($d \geq 1$):

$$P(\alpha) = \prod_{j=1}^{d+d_0} (\alpha - j), \quad \deg(P(\alpha)) = d + d_0,$$

for $k = d - 1$, ($d \geq 1$):

$$P(\alpha) = \prod_{j=2}^{d+d_0} (\alpha - j), \quad \deg(P(\alpha)) = d + d_0 - 1,$$

for $k = d - 2$, ($d \geq 2$):

$$P(\alpha) = \prod_{j=3}^{d+d_0} (\alpha - j), \quad \deg(P(\alpha)) = d + d_0 - 2,$$

and so on. Thus, from (6) we get that the factor with $k = d$ contributes to all the coefficients $a_0, a_1, \dots, a_{d+d_0}$ ($d \geq 1$), the factor with $k = d - 1$ to all from a_1 to a_{d+d_0} ($d \geq 1$), the factor with $k = d - 2$ to all from a_2 ($d \geq 2$), and so on. Obviously the j -th coefficient is constant iff each one of its factors (except the $k = d$ one) vanishes, in fact the term $\left(1 - \frac{\|w\|^2}{N_\Omega(z, \bar{z})^\mu}\right)$ in each factor has a different power.

In particular, the coefficient a_i , $i = 1, \dots, d + d_0$, contains the factor:

$$\frac{1}{\mu^d} \frac{D^{d-1} \tilde{\chi}(d)}{(d-1)!} \left(1 - \frac{\|w\|^2}{N_\Omega(z, z)^\mu}\right) A_i^2, \quad (d \geq 1),$$

and the coefficient a_i , $i = 2, \dots, d + d_0$, contains the factor:

$$\frac{1}{\mu^d} \frac{D^{d-2} \tilde{\chi}(d)}{(d-2)!} \left(1 - \frac{\|w\|^2}{N_\Omega(z, z)^\mu}\right)^2 A_{i-1}^3, \quad (d \geq 2),$$

where we denote by A_p^q the p -th coefficient of the polynomial in α :

$$\prod_{j=q}^{d+d_0} (\alpha - j).$$

Observe that A_i^2 and A_{i-1}^3 do not vanish. In fact we have:

$$\prod_{j=2}^{d+d_0} = \alpha^{d+d_0} + e_1(2, \dots, d+d_0) \alpha^{d+d_0-1} + \dots + e_{d+d_0}(2, \dots, d+d_0),$$

$$\prod_{j=3}^{d+d_0} = \alpha^{d+d_0-1} + e_1(3, \dots, d+d_0) \alpha^{d+d_0-2} + \dots + e_{d+d_0-1}(3, \dots, d+d_0)$$

where $e_j(x_1, \dots, x_n)$ is the elementary symmetric polynomial in the variables (x_1, \dots, x_n) , i.e.:

$$e_j(x_1, \dots, x_n) = \sum_{1 \leq k_1 < k_2 < \dots < k_j \leq n} x_{k_1} \dots x_{k_j}.$$

Since in our case x_j are positive integers, $A_i^2 = e_i(2, \dots, d+d_0)$ and $A_{i-1}^3 = e_{i-1}(3, \dots, d+d_0)$ do not vanish.

Thus we have that for $d \geq 2$ and for each $i = 3, \dots, d$, if a_i is constant then $D^{d-2} \tilde{\chi}(d) = D^{d-1} \tilde{\chi}(d) = 0$, and conclusion follows by [5], where in the proof of Theorem 1.3 it is pointed out that when $d > 1$ we have

$$D^{d-2} \tilde{\chi}(d) = D^{d-1} \tilde{\chi}(d) = 0,$$

if and only if a_2 is constant.

If $d = 1$, then $r = 1$ and $\Omega = \mathbb{C}H^1$, thus we need only to prove that if a_j is constant for some $j = 3, 4, \dots, d_0 + 1$, then $\mu = 1$. By the discussion above, if a_j is constant for some $j = 3, 4, \dots, d_0 + 1$ then $D^0 \tilde{\chi}(1) = 0$, which by [5, Lemma 3.5] directly implies $\mu = 1$, concluding the proof.

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